ON THE KERNEL OF THE GASSNER REPRESENTATION

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Abstract. We study the Gassner representation of the pure braid group P_n by considering its restriction to a free subgroup F. The kernel of the restriction is shown to lie in the subgroup $[\Gamma^3 F, \Gamma^2 F]$, sharpening a result of Lipschutz.

1. Introduction

Denote by $G_n : P_n \to GL_n(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ the unreduced Gassner representation of the pure braid group P_n (a formula is given in Section 2 below). The faithfulness of G_n for $n \geq 4$ is a long-standing open question. In this note, we investigate this by considering the restriction of G_n to a certain free subgroup F_{n-1} of P_n:

$$g_n : F_{n-1} \longrightarrow GL_n(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]).$$

The faithfulness of G_n would follow from that of g_n (see Proposition 2.1 below, or [3] for a more general result).

For a group H, denote by $\Gamma^\bullet H$ the lower central series of H. The main result of this paper is the following.

Theorem 3.4. *The kernel of g_n lies in the subgroup $[\Gamma^3 F_{n-1}, \Gamma^2 F_{n-1}]$.*

This is proved by passing to the graded quotients associated to the lower central series of F_{n-1} and the filtration of $GL_n(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ by powers of the augmentation ideal $J = \ker\{\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \xrightarrow{t_i \mapsto 1} \mathbb{Z}\}$. This allows us to show that the kernel of g_n lies in $\Gamma^5 F_{n-1}$. In [8], S. Lipschutz proved that the kernel of g_n lies in $[\Gamma^2 F_{n-1}, \Gamma^2 F_{n-1}]$ using different techniques (see also [1] for another proof). These two facts together allow us to prove Theorem 3.4.

We also show (Theorem 3.5) that the intersection of the kernel of g_n with $\Gamma^s F_{n-1}$ lies in the subgroup $[\Gamma^{s-2} F_{n-1}, \Gamma^2 F_{n-1}] \cdot \Gamma^{s+1} F_{n-1}$.

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2. Preliminaries and Notation

2.1. **The Gassner representation.** Denote by A_{rs}, $1 \leq r < s \leq n$, the generators of P_n. The (unreduced) Gassner representation is the homomorphism $G_n : P_n \to GL_n(\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ given by the formula:

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$$G_n(A_{rs}) = \begin{pmatrix} I_{r-1} & 0 & 0 & 0 & 0 \\ 0 & 1-t_r+t_rt_s & 0 & t_r(1-t_r) & 0 \\ 0 & \vec{u} & I_{s-r-1} & \vec{v} & 0 \\ 0 & 1-t_s & 0 & t_r & 0 \\ 0 & 0 & 0 & 0 & I_{n-s} \end{pmatrix}$$

where

$$\vec{u} = \big((1-t_{r+1})(1-t_s) \quad \cdots \quad (1-t_{s-1})(1-t_s)\big)^\top$$

and

$$\vec{v} = \big((1-t_{r+1})(t_r-1) \quad \cdots \quad (1-t_{s-1})(t_r-1)\big)^\top$$

and I_k denotes the $k \times k$ identity matrix. This representation is reducible to an $(n-1)$-dimensional representation, but the resulting formula is more complicated.

2.2. **The free subgroup.** Denote by F_{n-1} the free subgroup of P_n obtained by deleting the last string; this subgroup has generators $A_{1n}, A_{2n}, \ldots, A_{n-1,n}$. Moreover, we have a split short exact sequence

$$1 \longrightarrow F_{n-1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow 1$$

so that P_n is the semidirect product of P_{n-1} and F_{n-1}. Also, the following diagram commutes ([2], p. 138):

$$\begin{tikzcd} P_n \arrow[r] \arrow[d, "G_n"'] & P_{n-1} \arrow[d, "G_{n-1}"] \\ G_n(P_n) \arrow[r] & G_{n-1}(P_{n-1}) \end{tikzcd}$$

where the lower horizontal map is given by setting $t_n = 1$ and deleting the nth row and column.

Denote by $\Gamma^\bullet F_{n-1}$ the lower central series of F_{n-1} and for each i, consider the free abelian group

$$\Gamma^i F_{n-1}/\Gamma^{i+1} F_{n-1}.$$

We shall need an explicit basis of each $\Gamma^i F_{n-1}/\Gamma^{i+1} F_{n-1}$; this is given by the set of basic commutators of weight i. These are defined as follows. Denote by x_j the image of A_{jn} in $F_{n-1}/\Gamma^2 F_{n-1}$. Then the x_j are the basic commutators of weight one (denote this by $w(x_j) = 1$) and having defined the basic commutators of weight less than i, the basic commutators of weight i are the various $[c_u, c_v]$ where

(1) c_u and c_v are basic with $w(c_u) + w(c_v) = i$, and
(2) $c_u > c_v$ and if $c_u = [c_a, c_b]$, then $c_v \geq c_b$.

The commutators are ordered as follows. Those of weight i follow those of weight less than i and are ordered arbitrarily with respect to each other. A proof that the basic commutators of weight i form a basis of $\Gamma^i F_{n-1}/\Gamma^{i+1} F_{n-1}$ may be found in [4], p. 175.

Denote by g_n the restriction of G_n to F_{n-1} and set $X_n = g_n(F_{n-1})$.

Proposition 2.1.

$$\begin{aligned} G_n \textit{ is faithful} \quad &\Leftrightarrow \quad g_n \textit{ is faithful} \\ &\Leftrightarrow \quad \textit{the map } \Gamma^i F_{n-1}/\Gamma^{i+1} F_{n-1} \to \Gamma^i X_n/\Gamma^{i+1} X_n \\ &\qquad \textit{is injective for each } i \geq 1. \end{aligned}$$

Proof. A proof of the first equivalence may be found in, for example, [3]. The second equivalence is an easy exercise about free groups and is left to the reader. □

2.3. **The congruence subgroup.** Denote the subgroup of $GL_n(\mathbb{Z}[t_1^{\pm1}, \dots, t_n^{\pm1}])$ consisting of those matrices A with $A \equiv I_n$ modulo $(t_1 - 1, t_2 - 1, \dots, t_n - 1)$ by K_n (i.e., $K_n = GL_n(R, J)$ for $R = \mathbb{Z}[t_1^{\pm1}, \dots, t_n^{\pm1}]$ and J the augmentation ideal). Note that the image of P_n under G_n lies in K_n. The group K_n is filtered by powers of J:

$$K_n^i = \{A \in K_n : A \equiv I_n \mod J^i\}.$$

This is a central series: $[K_n^i, K_n^j] \subseteq K_n^{i+j}$.

Consider the graded quotients K_n^i/K_n^{i+1}. Note that $\Gamma^i K_n \subseteq K_n^i$, but it is probably not true that $K_n^\bullet$ is the lower central series (for $R = \mathbb{Z}[t, t^{-1}]$, the corresponding group K_n has $K_n^i/\Gamma^i K_n$ a torsion group for $n \geq 4$ [7]). Consider the induced map

$$\Phi^i : \Gamma^i F_{n-1}/\Gamma^{i+1} F_{n-1} \longrightarrow K_n^i/K_n^{i+1}.$$

Then by Proposition 2.1, we have the following:

$$G_n \text{ is injective if } \Phi^i \text{ is injective for all } i \geq 1.$$

We show in Section 3 that Φ^k is injective for $k \leq 4$, but that injectivity fails for $k = 5$.

2.4. **Structure of K_n^i/K_n^{i+1}.** Given $A \in K_n^i$, we may write

$$A \equiv I_n + \sum_{1 \leq \ell_1 \leq \cdots \leq \ell_i \leq n} (t_{\ell_1} - 1) \cdots (t_{\ell_i} - 1) A_{\ell_1, \dots, \ell_i} \mod J^{i+1},$$

where $A_{\ell_1, \dots, \ell_i} \in M_n(\mathbb{Z})$. Define

$$\pi_i : K_n^i \longrightarrow \bigoplus_{1 \leq \ell_1 \leq \cdots \leq \ell_i \leq n} M_n(\mathbb{Z})$$

by

$$\pi_i(A) = (A_{\ell_1, \dots, \ell_i})_{1 \leq \ell_1 \leq \cdots \leq \ell_i \leq n}.$$

π_i is clearly a homomorphism and $\ker \pi_i = K_n^{i+1}$.

Denote by $e_{\ell m}(a)$ the matrix having a in the ℓ, m position and zeroes elsewhere. Note that π_1 is surjective:

$$\pi_1(I_n + e_{\ell m}(t_j - 1)) = (0, \dots, 0, e_{\ell m}(1), 0, \dots, 0)$$

where $e_{\ell m}(1)$ appears in the summand corresponding to $(t_j - 1)$. Note that this works for $\ell = m$ as $1 + (t_j - 1) = t_j$ is a unit in $\mathbb{Z}[t_1^{\pm1}, \dots, t_n^{\pm1}]$. For $i > 1$, the image of π_i is the sum of copies of $M_n^0(\mathbb{Z})$ (matrices of trace 0):

$$\pi_i(I_n + e_{\ell m}((t_{j_1} - 1) \cdots (t_{j_i} - 1))) = (0, \dots, 0, e_{\ell m}(1), 0, \dots, 0)$$

occurring in the summand corresponding to the monomial $(t_{j_1} - 1) \cdots (t_{j_i} - 1)$ for $\ell \neq m$. Also, we can hit $e_{\ell,\ell}(1) - e_{\ell+1,\ell+1}(1)$ since

$$U = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 1 + (t_{j_1} - 1) \cdots (t_{j_i} - 1) & -(t_{j_1} - 1) \cdots (t_{j_i} - 1) & & \\ & & & (t_{j_1} - 1) \cdots (t_{j_i} - 1) & 1 - (t_{j_1} - 1) \cdots (t_{j_i} - 1) & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{bmatrix} \in K_n^i$$

and $\pi_i(U) = e_{\ell,\ell}(1) - e_{\ell+1,\ell+1}(1) + e_{\ell+1,\ell}(1) - e_{\ell,\ell+1}(1)$.

3. The Main Result

Let us now investigate the map $\Phi^i : \Gamma^i F_{n-1}/\Gamma^{i+1}F_{n-1} \to K_n^i/K_n^{i+1}$. When we refer to a "factor" we mean the copy of $M_n(\mathbb{Z})$ or $M_n^0(\mathbb{Z})$ in K_n^i/K_n^{i+1} corresponding to a certain monomial $(t_{j_1}-1)\cdots(t_{j_i}-1)$; we will abbreviate this monomial to $t_{j_1}\cdots t_{j_i}$.

Now, we have

$$\Phi^1(x_r) = \quad (\underset{t_n}{e_{rr}(1) + e_{nr}(-1)}, \quad \underset{t_r}{e_{rn}(-1) + e_{nn}(1)})$$

where the monomial under an entry indicates the factor in which it lies. For $1 \le r \le n-1$, these elements are clearly linearly independent in $K_n/K_n^2 = M_n(\mathbb{Z})$, and so Φ^1 is injective.

On the second level we have, for $r > s$,

$$\Phi^2([x_r, x_s]) = \quad (\underset{t_st_n}{e_{sr}(-1) + e_{nr}(1)}, \quad \underset{t_rt_n}{e_{rs}(1) + e_{ns}(-1)}, \quad \underset{t_rt_s}{e_{rn}(-1) + e_{sn}(1)})$$

and these are clearly linearly independent in K_n^2/K_n^3. So Φ^2 is injective as well.

To calculate Φ^3, we must order the bases of $F_{n-1}/\Gamma^2 F_{n-1}$ and $\Gamma^2 F_{n-1}/\Gamma^3 F_{n-1}$. Use the obvious order on the first level: $x_{n-1} > x_{n-2} > \cdots > x_1$. On the second level, use

$$[x_{n-1}, x_{n-2}] > \cdots > [x_{n-1}, x_1] > [x_{n-2}, x_{n-3}] > \cdots > [x_3, x_2] > [x_3, x_1] > [x_2, x_1].$$

Then a basis of $\Gamma^3 F_{n-1}/\Gamma^4 F_{n-1}$ is the set

$$[[x_r, x_s], x_u] \qquad r > s, u \ge s.$$

We have the following formula for $\Phi^3([[x_r, x_s], x_u])$:

factor	element
$t_st_n^2$	$e_{sr}(\delta_{us}) + e_{sr}(-\delta_{ur}) + e_{nr}(-\delta_{us}) + e_{nr}(\delta_{ur})$
$t_st_ut_n$	$e_{sn}(\delta_{ru}) + e_{ur}(1) + e_{nn}(-\delta_{ru}) + e_{nr}(-1)$
$t_rt_n^2$	$e_{rs}(\delta_{us}) + e_{rs}(-\delta_{ur}) + e_{ns}(\delta_{ur}) + e_{ns}(-\delta_{us})$
$t_rt_ut_n$	$e_{rn}(-\delta_{us}) + e_{us}(-1) + e_{nn}(\delta_{us}) + e_{ns}(1)$
$t_st_rt_n$	$e_{rn}(\delta_{ur}) + e_{ru}(1) + e_{nn}(-\delta_{ur}) + e_{sn}(-\delta_{us}) + e_{su}(-1) + e_{nn}(\delta_{us})$
$t_rt_st_u$	$e_{rn}(-1) + e_{sn}(1)$

Write c_{rsu} for $\Phi^3([[x_r, x_s], x_u])$.

Proposition 3.1. *$\{c_{rsu} : r > s, u \ge s\}$ is a linearly independent set in K_n^3/K_n^4.*

Proof. Suppose

$$\sum m_{rsu}c_{rsu} = 0$$

for some $m_{rsu} \in \mathbb{Z}$. If $s = u$, the factor $t_s^2 t_r = t_rt_st_u$ comes into play and does not occur in any other c_{rsu}. So $m_{rsu} = 0$ in this case. Similarly if $r = u$, the factor $t_r^2 t_s = t_rt_st_u$ comes into play and does not occur in any other c_{rsu} and so $m_{rsu} = 0$ here as well.

Thus, we may assume we have $\sum m_{rsu}c_{rsu} = 0$ where each of the c_{rsu} has distinct r, s, u. For a given fixed r, s, u, the factor $t_rt_st_u$ occurs exactly twice—in c_{rsu} and c_{usr}. The corresponding elements are $e_{rn}(-1) + e_{sn}(1)$ and $e_{un}(-1) + e_{sn}(1)$,

respectively. As $r \neq u$, these are linearly independent in $M_n(\mathbb{Z})$, and so we must have $m_{rsu} = m_{usr} = 0$ in this case as well. $\square$

Now let's look at Φ^4. A basis of $\Gamma^4 F_{n-1}/\Gamma^5 F_{n-1}$ consists of the elements

$$[[[x_r, x_s], x_u], x_v] \qquad r > s, u \geq s, v \geq u$$

and

$$[[x_r, x_s], [x_u, x_v]] \qquad r > s, u > v, r \geq u$$

(and in addition, if $r = u$, then $s > v$). Denote the image of an element above under Φ^4 by c_{rsuv}. Note that the order r, s, u, v uniquely determines which of the elements we have, as no sequence from the first type of basis element can occur as a sequence from the second type. Under Φ^4, the image of $[[[x_r, x_s], x_u], x_v]$ is

factor	element
$t_s t_n^3$	$e_{sv}(\delta_{us}\delta_{vr}) + e_{vr}(-\delta_{vs}) + e_{nr}(\delta_{su}\delta_{vs}) + e_{sv}(-\delta_{ur}\delta_{vr})$ $+ e_{vr}(\delta_{ur}\delta_{vs}) + e_{nr}(-\delta_{ur}\delta_{vs}) + e_{nv}(-\delta_{us}\delta_{vr}) + e_{nv}(-\delta_{ur}\delta_{vr})$
$t_s t_v t_n^2$	$e_{sn}(-\delta_{us}\delta_{rv}) + e_{sn}(\delta_{ur}\delta_{rv}) + e_{vr}(-\delta_{us}) + e_{nn}(\delta_{us}\delta_{rv})$ $+ e_{nr}(\delta_{us}) + e_{vr}(\delta_{ur}) + e_{nn}(-\delta_{ur}\delta_{rv}) + e_{nr}(-\delta_{ur})$
$t_u t_s t_n^2$	$e_{vn}(-\delta_{ru}\delta_{vs}) + e_{sv}(-\delta_{ru}) + e_{nn}(\delta_{ru}\delta_{sv}) + e_{uv}(\delta_{rv})$ $+ e_{vr}(-\delta_{uv}) + e_{nr}(\delta_{uv}) + e_{nv}(\delta_{ru}) + e_{nv}(-\delta_{rv})$
$t_u t_s t_v t_n$	$e_{sn}(\delta_{ru}) + e_{un}(-\delta_{rv}) + e_{vn}(-\delta_{ru})$ $+ e_{vr}(-1) + e_{nn}(\delta_{rv}) + e_{nr}(1)$
$t_r t_n^3$	$e_{rv}(\delta_{us}\delta_{vs}) + e_{vs}(-\delta_{us}\delta_{vr}) + e_{ns}(\delta_{us}\delta_{vr}) + e_{rv}(-\delta_{ur}\delta_{vs})$ $+ e_{vs}(\delta_{ur}\delta_{vr}) + e_{ns}(\delta_{ur}\delta_{vr}) + e_{nv}(\delta_{ur}\delta_{sv}) + e_{nv}(-\delta_{us}\delta_{vs})$
$t_r t_v t_n^2$	$e_{rn}(-\delta_{us}\delta_{vs}) + e_{rn}(\delta_{ur}\delta_{vs}) + e_{vs}(\delta_{ur}) + e_{nn}(-\delta_{ur}\delta_{vs})$ $+ e_{ns}(-\delta_{ur}) + e_{vs}(-\delta_{us}) + e_{nn}(\delta_{us}\delta_{vs}) + e_{ns}(\delta_{us})$
$t_r t_u t_n^2$	$e_{vn}(\delta_{us}\delta_{vr}) + e_{rv}(\delta_{us}) + e_{nn}(-\delta_{us}\delta_{vr}) + e_{uv}(-\delta_{vs})$ $+ e_{vs}(\delta_{uv}) + e_{ns}(-\delta_{uv}) + e_{nv}(-\delta_{us}) + e_{nv}(\delta_{vs})$
$t_r t_u t_v t_n$	$e_{rn}(-\delta_{us}) + e_{un}(\delta_{vs}) + e_{vn}(\delta_{us})$ $+ e_{vs}(1) + e_{nn}(-\delta_{vs}) + e_{ns}(-1)$
$t_r t_s t_n^2$	$e_{vn}(-\delta_{ur}\delta_{vr}) + e_{rv}(-\delta_{ur}) + e_{nn}(\delta_{ur}\delta_{vr}) + e_{rv}(\delta_{uv}) + e_{vu}(-\delta_{vr})$ $+ e_{nu}(\delta_{vr}) + e_{nv}(\delta_{ur}) + e_{vn}(\delta_{us}\delta_{vs}) + e_{sv}(\delta_{us}) + e_{nn}(-\delta_{us}\delta_{vs})$ $+ e_{sv}(-1) + e_{vu}(\delta_{vs}) + e_{nu}(-\delta_{vs}) + e_{nu}(-\delta_{vs}) + e_{nv}(-\delta_{vs})$
$t_r t_s t_v t_n$	$e_{rn}(\delta_{ur}) + e_{rn}(-\delta_{uv}) + e_{vn}(-\delta_{ur})$ $+ e_{sn}(-\delta_{us}) + e_{sn}(\delta_{uv}) + e_{vn}(\delta_{us})$
$t_r t_s t_u t_n$	$e_{vn}(\delta_{rv}) + e_{vn}(-\delta_{sv}) + e_{rv}(1)$ $+ e_{nn}(-\delta_{rv}) + e_{sv}(-1) + e_{nn}(\delta_{sv})$
$t_r t_s t_u t_v$	$e_{rn}(-1) + e_{sn}(1)$

and the image of $[[x_r, x_s], [x_u, x_v]]$ is

factor	element
$t_s t_v t_n^2$	$e_{su}(\delta_{rv}) + e_{vr}(-\delta_{us}) + e_{nr}(\delta_{us}) + e_{nu}(-\delta_{rv})$
$t_s t_u t_n^2$	$e_{sv}(-\delta_{ru}) + e_{ur}(\delta_{vs}) + e_{nr}(-\delta_{vs}) + e_{nv}(\delta_{ru})$
$t_s t_u t_v t_n$	$e_{sn}(\delta_{ru}) + e_{sn}(\delta_{rv}) + e_{ur}(1)$ $+ e_{nn}(-\delta_{ru}) + e_{vr}(-1) + e_{nn}(\delta_{rv})$
$t_r t_v t_n^2$	$e_{ru}(-\delta_{sv}) + e_{vs}(\delta_{ru}) + e_{ns}(-\delta_{ru}) + e_{nu}(\delta_{sv})$
$t_r t_u t_n^2$	$e_{rv}(\delta_{us}) + e_{us}(-\delta_{rv}) + e_{ns}(\delta_{rv}) + e_{nv}(-\delta_{su})$
$t_r t_u t_v t_n$	$e_{rn}(-\delta_{us}) + e_{rn}(\delta_{vs}) + e_{us}(-1)$ $+ e_{nn}(\delta_{us}) + e_{vs}(1) + e_{nn}(-\delta_{vs})$
$t_r t_s t_v t_n$	$e_{vn}(-\delta_{ru}) + e_{ru}(-1) + e_{nn}(\delta_{ru})$ $+ e_{vn}(\delta_{su}) + e_{su}(1) + e_{nn}(-\delta_{su})$
$t_r t_s t_u t_n$	$e_{un}(\delta_{rv}) + e_{rv}(1) + e_{nn}(-\delta_{rv})$ $+ e_{un}(-\delta_{sv}) + e_{sv}(-1) + e_{nn}(\delta_{sv})$
$t_r t_s t_u t_v$	0

Proposition 3.2. *The elements $\{c_{rsuv}\}$ are linearly independent in K_n^4/K_n^5.*

Proof. Suppose $\sum m_{rsuv} c_{rsuv} = 0$ for some $m_{rsuv} \in \mathbb{Z}$. Consider the case where only 2 of the r, s, u, v are distinct (e.g. c_{2111}, c_{3133}, etc.). Then c_{rsuv} contributes to the factor $t_r t_s t_u t_v = t_i^k t_j^\ell$ where i and j are the distinct indices and $1 \le k, \ell \le 3$, $k + \ell = 4$. Note that c_{rsuv} is the only contributor to this factor as the choice of r determines the sequence—the only possibilities are $rsrr$, $rsss$ or $rssr$ (note that no $[[x_r, x_s], [x_u, x_v]]$ occur as $r \neq s$, $u \neq v$ implies that $r = u$ and $s = v$ and so the element is 0). To this factor, c_{rsuv} contributes $e_{rn}(-1) + e_{sn}(1)$. It follows that $m_{rsuv} = 0$ for these elements.

Now suppose that $c_{rsuv} = \Phi([[[x_r, x_s], x_u], x_v])$ with r, s, u, v distinct (we shall deal with the double commutators with 4 distinct indices below). Then c_{rsuv} contributes the element $e_{rn}(-1) + e_{sn}(1)$ to the factor $t_r t_s t_u t_v$. For fixed r, s, u, v, we must have $r > s$ and $s < u < v$ if they are all distinct. So the only contributors to this factor are (1) c_{rsuv}, (2) c_{usrv} or c_{usvr} (the latter if $v < r$), (3) c_{vsru} or c_{vsur} (the latter if $u < r$); and they contribute $e_{rn}(-1) + e_{sn}(1)$, $e_{un}(-1) + e_{sn}(1)$, $e_{vn}(-1) + e_{sn}(1)$ respectively. Since these elements are linearly independent in $M_n(\mathbb{Z})$, we must have $m_{rsuv} = m_{usrv} = m_{vsru} = 0$.

Next, suppose that r, s, u, v consist of 3 distinct indices. We have $r > s$, $u \ge s$, and $v \ge u$. There are three cases to consider. Recall that in any case, $r > s$.

Case 1. $s = u$. We may assume (without loss of generality) that $r > v$ and $v > u$. Then we have three possible elements to consider: c_{rssv}, c_{vssr}, and c_{rsvs} (the latter corresponds to $[[x_r, x_s], [x_v, x_s]]$). Here, the factor $t_s^2 t_v t_r$ receives contributions only from c_{rssv} and c_{vssr}, the elements being $e_{rn}(-1) + e_{sn}(1)$ and $e_{vn}(-1) + e_{sn}(1)$, respectively. These are linearly independent in $M_n(\mathbb{Z})$ and so $m_{rssv} = m_{vssr} = 0$. Then, in the factor $t_s^2 t_r t_n$, the only contributors are c_{rssv}, c_{vssr} and c_{rsvs}—the latter contributing $e_{rv}(1) + e_{sv}(-1)$, while the former two contribute $e_{rv}(1) + e_{sv}(-1)$ and $e_{vr}(1) + e_{sr}(-1)$. As we've already shown that $m_{rssv} = m_{vssr} = 0$, and since the only other c_{rsuv} that contribute here only have two distinct indices, we must have $m_{rsvs} = 0$ as well.

Case 2. $u = v$. Then $u > s$. Suppose $r > u$ (the case $r < u$ is Case 3 below). Then the three elements to consider are c_{rsuu}, c_{usur} and c_{ruus}. The first two elements

contribute to the factor $t_s t_u^2 t_r$ the elements $e_{rn}(-1) + e_{sn}(1)$ and $e_{un}(-1) + e_{sn}(1)$ respectively and no other c_{ijkl} contributes to this factor. So $m_{rsuu} = m_{usur} = 0$. Then consider the factor $t_u^2 t_r t_n$. Here, c_{ruus} contributes $e_{ru}(1) + e_{su}(-1)$ and the only other contributors have $m_{ijkl} = 0$ already. Thus, $m_{ruus} = 0$, as well.

Case 3. $u = v$, $u > r$. This is similar to Case 2.

Finally, consider the $c_{rsuv} = \Phi([[x_r, x_s], [x_u, x_v]])$ with r, s, u, v distinct. Then $r > s$, $u > v$, $r > u$. As the indices are distinct the only factors contributed to are $t_s t_u t_v t_n$, $t_r t_u t_v t_n$, $t_r t_s t_v t_n$, and $t_r t_s t_u t_n$ (see the formulas above). For fixed r, s, u, v, the only possible c_{ijkl} are given in the following table, along with the elements contributed to each factor.

	$t_s t_u t_v t_n$	$t_r t_u t_v t_n$	$t_r t_s t_v t_n$	$t_r t_s t_u t_n$
c_{rsuv}	$e_{ur}(1) + e_{vr}(-1)$	$e_{us}(-1) + e_{vs}(1)$	$e_{ru}(-1) + e_{su}(1)$	$e_{rv}(1) + e_{sv}(-1)$
c_{rusv}	$e_{sr}(1) + e_{vr}(-1)$	$e_{rs}(-1) + e_{us}(1)$	$e_{su}(-1) + e_{vu}(1)$	$e_{rv}(1) + e_{uv}(-1)$
c_{rvsu}	$e_{sr}(1) + e_{ur}(-1)$	$e_{rs}(-1) + e_{vs}(1)$	$e_{ru}(1) + e_{vu}(-1)$	$e_{sv}(-1) + e_{uv}(1)$

In each factor, we obtain linearly dependent elements, but we must remember that we're scaling the element coming from c_{rsuv} by m_{rsuv}. Looking at the factor $t_s t_u t_v t_n$, we find that $m_{rsuv} = m_{rvsu}$ and $m_{rusv} = -m_{rvsu}$. But then looking at the factor $t_r t_u t_v t_n$, we find $m_{rsuv} = -m_{rvsu}$ and $m_{rusv} = -m_{rvsu}$. Thus, $m_{rsuv} = m_{rusv} = m_{rvsu} = 0$.

This completes the proof. □

Corollary 3.3. *The kernel of g_n is contained in $\Gamma^5 F_{n-1}$.* □

It is possible to sharpen Corollary 3.3 to obtain the main result.

Theorem 3.4. *The kernel of g_n is contained in $[\Gamma^3 F_{n-1}, \Gamma^2 F_{n-1}]$.*

Proof. For simplicity, denote the group $\Gamma^i F_{n-1}$ by Γ^i. By Corollary 3.3 and by [8], we have $\ker(g_n) \subseteq \Gamma^5 \cap [\Gamma^2, \Gamma^2]$. We claim that the latter group equals $[\Gamma^3, \Gamma^2]$. The main theorem in [6] implies that

$$\Gamma^5 \cap [\Gamma^2, \Gamma^2] = I_{\Gamma^2}([\Gamma^3, \Gamma^2]),$$

where $I_R(S)$ is the isolator of S in R. (Recall that the isolator of S in R is the set $I_R(S) = \{x \in R : x^n \in S \text{ for some } n\}$.) To see that this latter group is simply $[\Gamma^3, \Gamma^2]$, it suffices to show that the quotient group $\Gamma^2/[\Gamma^3, \Gamma^2]$ is torsion-free. Consider the short exact sequence

$$1 \longrightarrow \frac{[\Gamma^2, \Gamma^2]}{[\Gamma^3, \Gamma^2]} \longrightarrow \frac{\Gamma^2}{[\Gamma^3, \Gamma^2]} \longrightarrow \frac{\Gamma^2}{[\Gamma^2, \Gamma^2]} \longrightarrow 1.$$

Since Γ^2 is a free group, the group $\Gamma^2/[\Gamma^2, \Gamma^2]$ is free abelian. But, by Theorem 6 of [5], the group $[\Gamma^2, \Gamma^2]/[\Gamma^3, \Gamma^2]$ is also free abelian. It follows that $\Gamma^2/[\Gamma^3, \Gamma^2]$ is torsion-free. This completes the proof. □

The methods used above allow us to prove the following result.

Theorem 3.5. *For $s \geq 5$, $\ker(g_n) \cap \Gamma^s \subseteq [\Gamma^{s-2}, \Gamma^2] \cdot \Gamma^{s+1}$.*

Proof. Note that any basic commutator is given by a unique list of integers corresponding to the x_j that occur in the commutator. For example, $[[x_3, x_2], [x_3, x_1]]$ yields the list $3, 2, 3, 1$. We therefore may denote a basic commutator of weight s by $x_{\ell_1 \ell_2 \cdots \ell_s}$ without confusion. Denote the element $\Phi^s(x_{\ell_1 \ell_2 \cdots \ell_s})$ by $c_{\ell_1 \ell_2 \cdots \ell_s}$. Note

that all the basic commutators in Γ^s/Γ^{s+1} lie in $[\Gamma^{s-2},\Gamma^2]$, except for the various $[c_u,x_j]$. Moreover, any element of the latter form must be an s-fold commutator:

$$x_{\ell_1\ell_2\cdots\ell_s}=[\cdots[[[x_{\ell_1},x_{\ell_2}],x_{\ell_3}],\cdots],x_{\ell_s}].$$

To prove the theorem, it suffices to show that if we have a dependency relation

$$\sum m_{\ell_1\ell_2\cdots\ell_s}c_{\ell_1\ell_2\cdots\ell_s}=0$$

where the $m_{\ell_1\cdots\ell_s}\in\mathbb{Z}$, then we have $m_{\ell_1\cdots\ell_s}=0$ whenever $x_{\ell_1\cdots\ell_s}$ is an s-fold commutator. This will show that the $[c_u,x_j]$ inject into K_n^s/K_n^{s+1} and hence that the intersection of the kernel of g_n with Γ^s lies in $[\Gamma^{s-2},\Gamma^2]\cdot\Gamma^{s+1}$.

Observe that in the case of an s-fold commutator, the element $c_{\ell_1\cdots\ell_s}$ contributes the element $e_{\ell_1,n}(-1)+e_{\ell_2,n}(1)$ to the factor $t_{\ell_1}t_{\ell_2}\cdots t_{\ell_s}$ (this is easily proved by induction using the formulas given above for the c_j and c_{rs}). Suppose the $\ell_1,\ell_2,\ldots,\ell_s$ consist of i distinct indices, say $r_1=\ell_1$, $r_2=\ell_2$, and $r_3,\ldots,r_i$. We have $r_1>r_2$ and $r_2<r_3<\cdots<r_i$. We have several contributors to the factor $t_{\ell_1}t_{\ell_2}\cdots t_{\ell_s}=t_{r_1}^{a_1}t_{r_2}^{a_2}\cdots t_{r_i}^{a_i}$ (here a_k is the number of times r_k occurs). Let us abbreviate notation and write $c_{r_1r_2\cdots r_i}$ for $c_{\ell_1\cdots\ell_s}$. Certain permutations of the ℓ_j yield s-fold basic commutators; each of these contributes to the factor $t_{r_1}^{a_1}t_{r_2}^{a_2}\cdots t_{r_i}^{a_i}$ under consideration. We must show that the resulting contributions are linearly independent. Note that the only contributors to this factor are s-fold commutators—if $c=[c_u,c_v]$ where $w(c_v)\geq 2$, then every factor to which c contributes contains a power of t_n (see the formulas above). Thus, we may detect any dependency relation among the $c_{\ell_1,\ldots,\ell_s}$ by considering only the factor $t_{\ell_1}\cdots t_{\ell_s}$.

Now, for some q with $3\leq q\leq i$ we must have $r_q<r_1<r_{q+1}$. Then we get the following contributions to the factor $t_{r_1}^{a_1}t_{r_2}^{a_2}\cdots t_{r_i}^{a_i}$:

sequence	element
$r_1,r_2,\ldots,r_i$	$e_{r_1,n}(-1)+e_{r_2,n}(1)$
$r_q,r_2,\ldots,r_{q-1},r_1,r_{q+1},\ldots,r_i$	$e_{r_q,n}(-1)+e_{r_2,n}(1)$
$r_{q+1},r_2,\ldots,r_q,r_1,r_{q+2},\cdots,r_i$	$e_{r_{q+1},n}(-1)+e_{r_2,n}(1)$
$r_{q+2},r_2,\ldots,r_q,r_1,r_{q+1},r_{q+3},\ldots,r_i$	$e_{r_{q+2},n}(-1)+e_{r_2,n}(1)$
$\vdots$	$\vdots$
$r_i,r_2,\ldots,r_q,r_1,r_{q+1},\ldots,r_{i-1}$	$e_{r_i,n}(-1)+e_{r_2,n}(1)$

Since $r_1,r_2,\ldots,r_i$ are distinct, the elements in the second column are linearly independent in $M_n(\mathbb{Z})$ (as $i<n$) and so each of the corresponding coefficients satisfies $m_{\ell_1\cdots\ell_s}=0$. This completes the proof. $\square$

4. Breakdown

The method used in Section 3 breaks down at the fifth level, however. Indeed, if $n=4$, the kernel of Φ^5 is rather large. For example, we have $c_{21131}=c_{31121}$. This has the interpretation that the degree 5 part of the polynomials in the Gassner matrices of

$$[[[A_{24},A_{14}],A_{14}],[A_{34},A_{14}]]$$

and

$$[[[A_{34},A_{14}],A_{14}],[A_{24},A_{14}]]$$

are the same. The matrices are not the same, however, and a computer search by the author based on the relations in the kernel of Φ^5 has not turned up any elements in the kernel of G_4.

Note, however, that the failure of the method does not imply that g_n is not injective. Really, one needs to consider the quotients $\Gamma^i g_n(F_{n-1})/\Gamma^{i+1}g_n(F_{n-1})$ rather than the classes of the various elements in $\Gamma^i g_n(F_{n-1})$ modulo the subgroup K_n^{i+1}. This seems to be rather intractable, however, given the ranks of the various $\Gamma^i F_{n-1}/\Gamma^{i+1}F_{n-1}$ (for example, $\Gamma^5 F_3/\Gamma^6 F_3$ has rank 116).

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